

Hydrodynamic Supercontinuum

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We demonstrate experimentally multi-bound-soliton solutions of the Nonlinear Schrödinger equation (NLS) in the context of surface gravity waves. In particular, the Satsuma-Yajima N -soliton solution with $N = 2, 3, 4$ is investigated in detail. Such solutions, also known as breathers on zero background, lead to periodic self-focussing in the wave group dynamics, and the consequent generation of a steep localized carrier wave underneath the group envelope. Our experimental results are compared with predictions from the NLS for low steepness initial conditions where wave-breaking does not occur, with very good agreement. We also show the first detailed experimental study of irreversible massive spectral broadening of the water wave spectrum, which we refer to by analogy with optics as the first controlled observation of hydrodynamic supercontinuum a process which is shown to be associated with the fission of the initial multi-soliton bound state into individual fundamental solitons similar to what has been observe in optics.

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The generation of new frequency components is a defining feature of nonlinear physics. Perhaps the most spectacular phenomenon of nonlinear physics occurs when a narrow band input wave group undergoes rapid spectral broadening as a result of strong nonlinear interactions to create a broadband spectrum. Such spectral broadening has been especially studied in an optical context where the long interaction length between an intense electromagnetic pulse and a nonlinear medium (solid, gases or waveguides) leads to the generation of a quasi-continuous broadband spectrum commonly referred to as supercontinuum (see for example [1]). The progress in the understanding of the complex mechanisms underlying the generation of a supercontinuum can be attributed to the nonlinear Schrödinger equation (NLS) model [2] and its extension to higher order terms that have been shown to explain all the essential physics behind the massive spectral broadening [1].

An important aspect of the NLS (with appropriate coefficients of course) is that it is ubiquitous and can be used to describe the dynamics of many nonlinear systems including for example besides the propagation of optical waves that of Langmuir waves in an unmagnetized plasma, ocean waves and, more generally, of all weakly nonlinear, narrow band dispersive waves [3]. For the surface gravity wave problem, the NLS equation has been derived more than 40 years ago [4, 5] and it is integrable via the Inverse Scattering Transform [2]. A variety of exact solutions have been presented during the last 40 years but the most celebrated is the propagation-invariant soliton which have been observed experimentally in the early 80's [6]. More recently, breather solutions of the NLS

have attracted significant attention as it has been suggested that *breathers on finite background* [7–11] can be considered as prototypes of rogue waves in the ocean [12–14]. Those waves with extreme amplitudes have been reproduced under controlled laboratory experiments in various systems ranging from hydrodynamic [15, 16] to fiber optics [17, 18] and plasma physics [19].

A second class of NLS breather solutions, known as *breather on zero background* (or sometimes higher-order solitons or Stsuma-Yajima breathers) [20–22], has received less attention, in particular in the water wave context. Moreover, to our knowledge, these class of solutions has never been observed in wave tank laboratory experiments. Higher-order solitons can be considered as the nonlinear superposition of several solitons where the relative-phase of the individual consituent solitons leads to recurrent cycles of pulse compression and subsequent recovery of the original pulse shape after a characteristic propagation distance (soliton period). The initial stage of propagation of a higher-order soliton is associated with temporal compression and local amplification of the amplitude, corresponding to broadening in the spectral domain. Importantly, the dynamics of higher-order soliton have been shown to play a central role in the generation of an optical supercontinuum when fission of the higher-order soliton into multiple fundamental solitons occurs as a result of perturbation to the NLS [1] (see also [23]).

In this Letter, we present the first experimental observations of the Satsuma-Yajima breathers on a zero background in a water wave tank and show that in a regime of strong nonlinearity the breathers can split into fundamental solitons resulting in the generation of a broad and

continuous spectrum which we describe as an hydronamic supercontinuum by analogy with the corresponding phenomenon in optics. Significantly, the fact that we are able to qualitatively interpret the features of the supercontinuum in terms of specific solutions of the NLS is similar to the way in which continuous spectral broadening has been analysed in optics. Our observations show clearly that the essential physics of water wave propagation remains well-described by the NLS over a wide range of experimental parameters.

Our analysis is based on the focusing NLS equation which is appropriate for describing the evolution of deep-water wave packets in space x [24]:

$$i \left(\frac{\partial A}{\partial x} + \frac{2k_0}{\omega_0} \frac{\partial A}{\partial t} \right) - \frac{k_0}{\omega_0^2} \frac{\partial^2 A}{\partial t^2} - k_0^3 |A|^2 A = 0. \quad (1)$$

Here k_0 represents the carrier wave number and ω_0 is the corresponding angular frequency related to k_0 *via* the dispersion relation $\omega_0 = \sqrt{gk_0}$, where g denotes the gravitational acceleration. Deep-water wave packets propagate with the group velocity: $c_g = \left. \frac{d\omega}{dk} \right|_{k=k_0} = \frac{\omega_0}{2k_0}$.

The surface elevation $\eta(x, t)$, taking into account the second Stokes harmonic, can be represented in terms of the complex envelope $A(x, t)$ as:

$$\eta(x, t) = \frac{1}{2} \left(A(x, t) e^{i\vartheta} + \frac{1}{2} k_0 A^2(x, t) e^{2i\vartheta} + c.c. \right), \quad (2)$$

where *c.c.* denotes complex conjugate and $\vartheta = (k_0 x - \omega_0 t)$. After rescaling space, time and amplitude in the equation (1), one obtains the well-known dimensionless form of the NLS:

$$i\psi_X + \psi_{TT} + 2|\psi|^2\psi = 0. \quad (3)$$

The two-soliton solution, which corresponds to a breather on zero background of Eq. (3) can be written analytically as [20]:

$$\psi_2(X, T) = \frac{4 (\cosh(3T) + 3 \cosh(T) \exp(8iX))}{\cosh(4T) + 4 \cosh(2T) + 3 \cos(8X)} \times \exp(iX), \quad (4)$$

Let the starting coordinate of the wave tank corresponds to $X = 0$ at the position of the mechanical paddle generating the waves. Then the initial condition at $X = 0$ for exciting this particular breather solution takes the simple hyperbolic-secant form with doubled amplitude $2\text{sech}(T)$. Higher order solutions can be obtained (for example) using the Darboux-transformation [21, 25]. This method allows us to write multi-soliton solutions in general form, although the exact expressions of higher-order solutions is cumbersome and not reproduced here. However, the initial condition for exciting the N -soliton solution remains simple with the generic form $\psi(X = 0, T) = N \text{sech}(T)$.

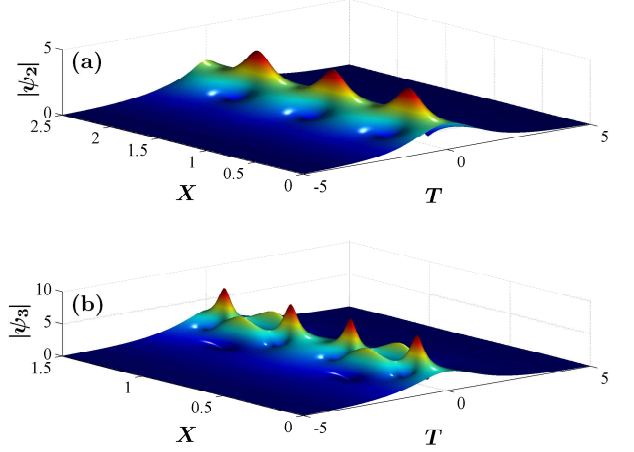


FIG. 1: (Color online)(a) Two-soliton and (b) three-soliton Satsuma-Yajima breathers.

Fig. 1 shows the evolution of the two-soliton and three-soliton Satsuma-Yajima solutions along X and T . Excitation of these breathers in a water tank is an important confirmation of the ability of the NLS to capture extreme localisation of water waves with high amplitude features. Yet, we should emphasize that the NLS model is only the lowest order approximation and the dynamics of the surface waves may be influenced by higher order dispersive and time-delay effects which can dramatically increase the spectral broadening during the evolution of multi-soliton solutions.

Our experimental setup is described in [16]. The initial condition generated by the wave maker is programmed according to Eq. (2). Each particular soliton solution requires a special choice of the carrier parameters. The initial steepness of the carrier, defined as $\varepsilon_0 = a_0 k_0$, plays a key role in the experiment with higher steepness ε_0 yielding more rapidly evolving dynamics. On the other hand, wave breaking defines a threshold steepness value beyond which the excited soliton solution will break before reaching its maximal amplitude. Breaking of the two-soliton and the three-soliton starts at initial steepnesses of 0.10 and 0.05, respectively. Thus, we kept the steepnesses below these values.

One should also keep in mind that the experiments are limited to the size of the tank and the initial conditions must therefore be properly scaled so as to capture the evolution dynamics and in particular the stage of maximal amplification of the excited breather solution. The experimental results showing the evolution of the two- and three-soliton are shown in the upper and lower panel of Fig. 2, respectively. The upper panel shows the evolution of the two-soliton solution for a carrier amplitude of 5 mm and a steepness value of 0.08. The lower panel shows the three-soliton solution for a carrier amplitude

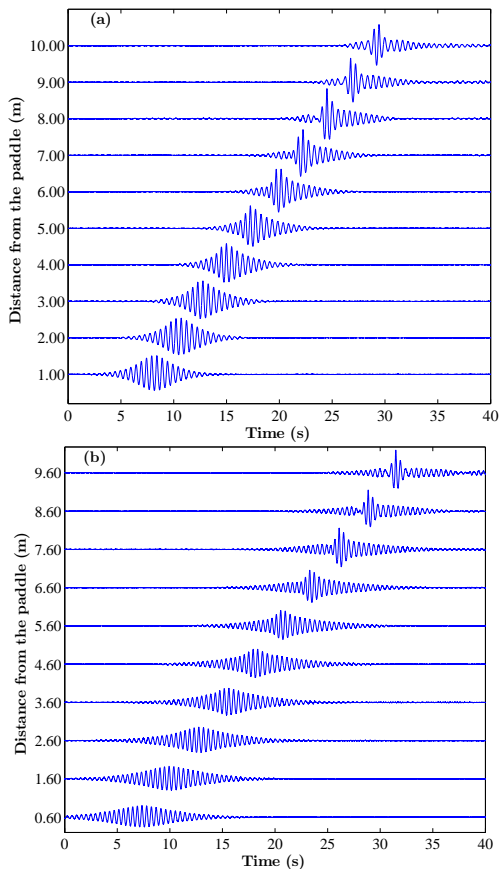


FIG. 2: (Color online) (a) Evolution of the two-soliton periodic breather along the tank for the carrier amplitude of $a_0 = 5$ mm and a carrier steepness of $\varepsilon = 0.08$. (b) Evolution of the three-soliton periodic breather along the tank for the carrier amplitude of $a_0 = 2$ mm and a carrier steepness of $\varepsilon = 0.04$.

of 2 mm and a steepness value of 0.04.

The initial carrier amplitude and steepness are chosen in order to observe the maximal amplification at the distances 8 m and 9.5 m from the paddle for the $N = 2$ and $N = 3$ soliton, respectively. Specifically, we used initial carrier amplitudes $a_0 = 5$ mm and $a_0 = 2$ mm for the second and third order soliton. Note also that by choosing a steepness values of 0.08 and 0.04 for the two- and three-soliton solutions, respectively, we are still far from the breaking limit. As predicted by the NLS, the initially $N\text{sech}(T)$ -shaped wave packet follows an initial stage of self-compression and until it reaches its maximal amplitude precisely at the expected spatial location and subsequent temporal broadening.

In order to confirm the validity of the observations, we also compare the experimentally observed wave groups at their maximal amplitude with the analytical higher order soliton solutions of the NLS. To this end, Fig. 3 shows the measured profile (blue line) together with the corresponding analytical soliton solution (red line) at the spatial coordinate of maximal amplitude and excellent

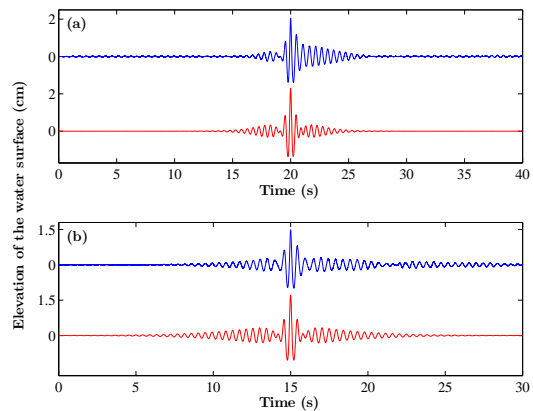


FIG. 3: (Color online) Comparison of the observations (upper blue curves) with corresponding NLS predictions (lower red curves) at the point of maximal wave amplification for: (a) the two-soliton breather with carrier parameters $a_0 = 5$ mm, $\varepsilon = 0.08$ and (b) the three-soliton breather with carrier parameters $a_0 = 2$ mm, $\varepsilon = 0.04$.

agreement is indeed observed for both the second and third order soliton. However, we can also see discrepancies in the form of a slight asymmetry in the experimental data. Such asymmetry can in fact be expected from the symmetry breaking of the NLS as induced by higher order dispersive and time-delay terms in the NLS equation [26] and which are not included in the pure NLS. The spectral broadening associated with the temporal compression of the higher order solitons evolution is highlighted in Fig. 4 where we plot both the theoretical and experimental spectral evolution of the $N = 2$ and $N = 3$ solitons. The experimental spectra were calculated from the observed wave envelopes using the Hilbert-transform every 20 cm along the tank and again we see excellent agreement with the analytical predictions.

Clearly, significant broadening of the spectra is observed during the evolution along the flume. As previously mentioned, the asymmetry of the experimental spectra is due to higher-order effects missing in the NLS approach. The remarkable broadening of the spectrum, related to the generation of supercontinuum, can be also observed directly from the measured spectra. Fig. 5 shows the broadening at the carrier peak frequency f_0 as well as at two higher harmonics $2f_0$ and $3f_0$. The latter correspond to the Stokes contributions always present in water waves.

In an optical context, the propagation N -bound-soliton solutions can lead to the generation of a broad supercontinuum when perturbation arising from higher order dispersion becomes important and split the initial bound state into fundamental individual solitons of different amplitudes and duration that separate with propagation, a mechanism generally referred to as soliton-fission [1, 27]. We next proceeded to demonstrate that the very same phenomenon can also manifest in hydrody-

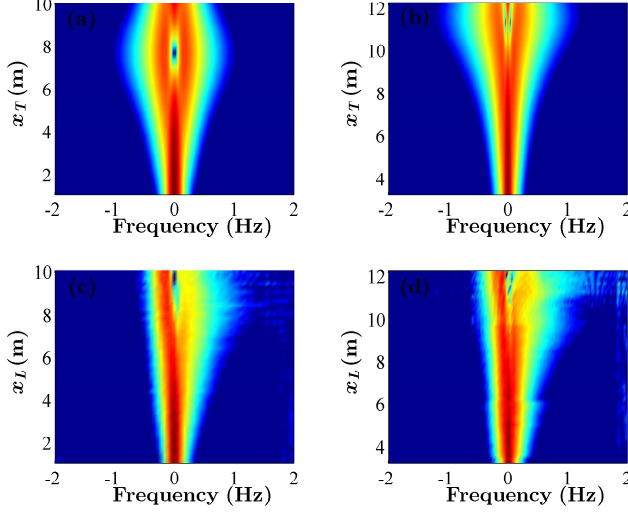


FIG. 4: (Color online) Two clear observations of the supercontinuum generation and their comparison with the theory. Two upper panels (a) & (b) show the theoretical spectra of the two-soliton and three-soliton pulsating solutions, respectively, propagating along the flume. (c) & (d) Corresponding experimental spectra calculated from measured envelopes using the Hilbert-transform.

namics leading to the generation of a water wave supercontinuum. For this purpose, we increase the nonlinearity in the dynamical system in order to increase the spectral bandwidth of the wave group at the stage of maximum compression so that higher order perturbations to the NLS (e.g. dispersion and time-delay) becomes significant and break the initial bound-state. The symmetry breaking caused by the higher-order terms thus results in soliton fission with the individual solitons travelling at different velocities and eventually spreading across the wave group just as in the optical case. In hydrodynamic propagation increasing the nonlinearity can be achieved either by increasing the order of the launched multi-soliton or by increasing the carrier-steepness. We favoured the former in our experiments because increasing the steepness can cause the initial wave to break. Launching a fourth order soliton ($4\text{sech}(T)$) into the tank with a carrier-amplitude $a_0 = 1$ mm and a carrier-steepness value of $\varepsilon = 0.04$ fsignature of the bound state fission into four distinct solitons is clearly identified in Fig. 6(a-b) which shows the temporal amplitude of the wave group at the beginning and end of the tank. The result is a quasi-continuous spectrum at the end of the wave tank as shown in Fig. 6(c).

In conclusion, we have presented the first observations of multi-soliton breathers on zero background in hydrodynamics. The measured maximal wave amplitudes are in very good agreement with the analytical solutions of the NLS and residual discrepancies are due to the higher-

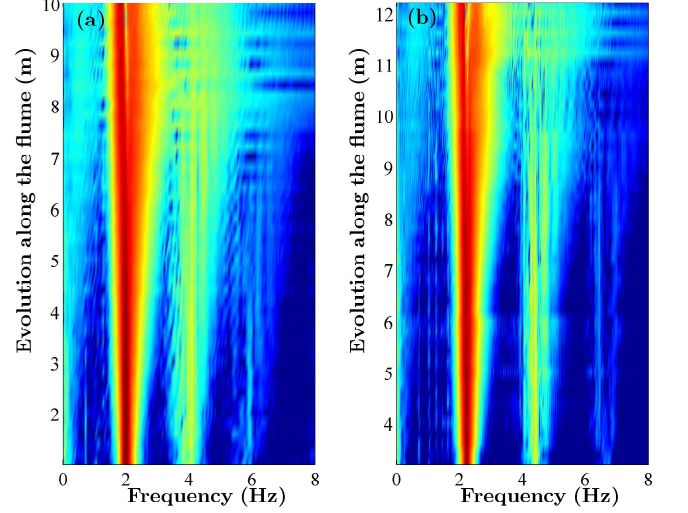


FIG. 5: (Color online) Evolution of spectra along the flume computed directly from the surface elevation data for the (a) two-soliton solution with carrier peak frequency $f_0 = 1.99$ Hz and (b) three-soliton breather with carrier peak frequency $f_0 = 2.22$ Hz. The spectral broadening is also observed at higher-Stokes components, that is, at $2f_0$ and $3f_0$ which can be expected for the water waves.

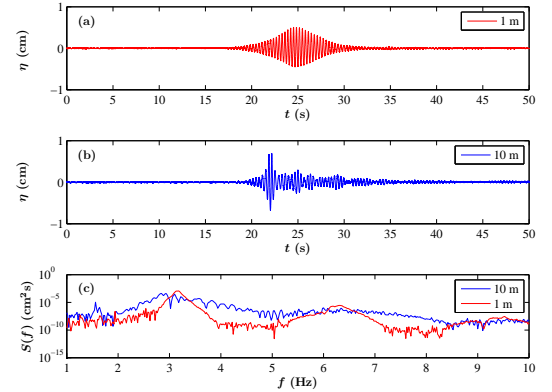


FIG. 6: (Color online) Fission of the four-soliton solution with initial condition $4\text{sech}(T)$ at $x = 0$ for the carrier parameters $a_0 = 1$ mm and $\varepsilon = 0.04$. (a) Wave-gauge measurement of the surface elevation at a position 1 m from the flap. (b) Wave-gauge measurement of the surface elevation at a position 9 m from the flap. (c) Measured spectra of the same data at 9 m, confirming the generation of supercontinuum.

order effects which are not taken into account in the weakly-nonlinear approach described by the NLS. When the nonlinearity of the system is increased, the massive spectral broadening seen during the evolution in the flume is associated with clear signatures of soliton fission, leading to what we believe is the first observation of supercontinuum in water waves. Our results reveal yet another correspondence between the dynamics of 1D wave

tanks and fiber-optic systems. We anticipate that these results will motivate further studies in hydrodynamics to observe nonlinear interactions of surface gravity waves but also in other nonlinear systems governed by similar type of NLS equations including plasma, acoustic or matter waves.

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